Introduction: Coming Attractions

Fundamental Concepts in Complex Analysis

Lecture Notes

September 8, 2026

1 Fundamental Theorem of Algebra

On the invention of Complex numbers:

Consider a cubic equation of the form $x^3 + px + q = 0$. Using the famous Cardano's formula (from 1545):

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Gerolamo Cardano

Is there only one root? No, there are always three complex cubic roots!

Example: Take the equation $x^3 - 15x = 4$, which means $p = -15$ and $q = -4$. We know a real root is $x = 4$. However, applying Cardano's formula gives:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What is wrong here? We need to take all possible values of the cubic roots to find the real solution.

> **The Fundamental Theorem of Algebra**
>
> Every (non-constant) polynomial with complex coefficients has a root. In fact, if it has degree n, it has at least n roots, counting multiplicity.

We will have at least three proofs of this Theorem!

2 Differentiation: Real vs. Complex

There are profound differences between differentiability in the real numbers and the complex numbers.

The Real Case

- **Definition:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Limited Differentiability:** A function can be differentiable only once. Example: $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

- **Taylor Series Disconnect:** Even infinitely differentiable functions can have nothing to do with their Taylor series. Example: $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Here $f^{(n)}(0) = 0 \ \forall n$.

- **Analyticity:** Real analytic functions equal their Taylor series: $f(x) = \sum a_n (x - x_0)^n$.

Hierarchy: Differentiable $\not\Rightarrow$ Infinitely Differentiable $\not\Rightarrow$ Analytic

The Complex Case

- **Definition:**

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Functions satisfying this are **analytic**.

- **Infinite Differentiability:** Any function differentiable at a neighborhood of a point is automatically *infinitely differentiable*.

- **Taylor Series Equivalence:** Differentiable functions are automatically equal to the sum of their Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Hierarchy: Differentiable $\implies$ Infinitely Differentiable $\implies$ Analytic

3 Series and Singularities

The behavior of power series highlights the necessity of the complex plane to understand singularities.

- **Real Singularity:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

This function has an obvious singularity at $x = 1$.

- **Complex Singularity:**

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

While there are *no real singularities* for this function, the series still has a restricted radius of convergence because of **complex singularities** at $z = \pm i$.

4 Antiderivatives: Real vs. Complex

The Real Case

Any continuous function has an antiderivative: $F'(x) = f(x)$. By the Fundamental Theorem of Calculus (FTC):

$$F(x) = \int_a^x f(t)dt$$

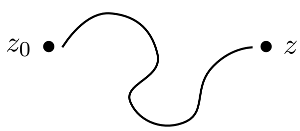
over which path?

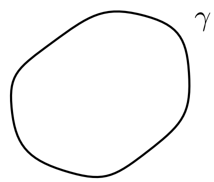
The Complex Case

Only differentiable functions have antiderivatives.

$$F(z) = \int_{z_0}^{z} f(w)dw$$

This is a line integral. Over which path? For a valid antiderivative, it must be *path-independent:* over a closed loop γ: $\oint_\gamma f(z)dz = 0$.





$$\oint_\gamma f(z)\,dz = 0$$

Real Differentiability in $\mathbb{R}^2$ vs. Complex Differentiability

Real differentiability of f at z (viewing $f : \mathbb{R}^2 \to \mathbb{R}^2$) means

$$\left|f(z+h) - f(z) - T(h)\right| \;\longrightarrow\; 0 \quad (\text{relative to } |h|),\text{ as } h \to 0,$$

for *some* $\mathbb{R}$-linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ – i.e. the Jacobian of f, with no further constraint on its four real entries. Writing $f = u + iv$, $z = x + iy$, this Jacobian is the 2×2 real matrix

$$T = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix},$$

four independent real numbers – generically, T need not have any special structure.

Complex differentiability asks for the *same* linear approximation, but with T forced to be **multiplication by a single complex number** λ:

$$\frac{\left|f(z+h) - f(z) - \lambda h\right|}{|h|} \;\longrightarrow\; 0, \qquad \lambda = f'(z).$$

Maps $h \mapsto \lambda h$ (rotation-scalings) are a very rigid 2-parameter family sitting inside all 4-parameter real-linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. Demanding the derivative take this restricted form *is* the Cauchy–Riemann equations – and this extra rigidity is exactly what forces analytic functions to be infinitely differentiable.

5 Conformal Maps and Riemann Theorem

Bernhard Riemann

Conformal Maps: A special and highly important class consists of injective analytic functions. A map $\varphi : \Omega_1 \to \Omega_2$ is conformal if it is an angle- and orientation-preserving map.

Riemann Mapping Theorem

If Ω is a simply-connected domain (a domain without holes), and $\Omega \neq \mathbb{C}$, then there exists a conformal bijection between $\mathbb{D}$ and Ω.